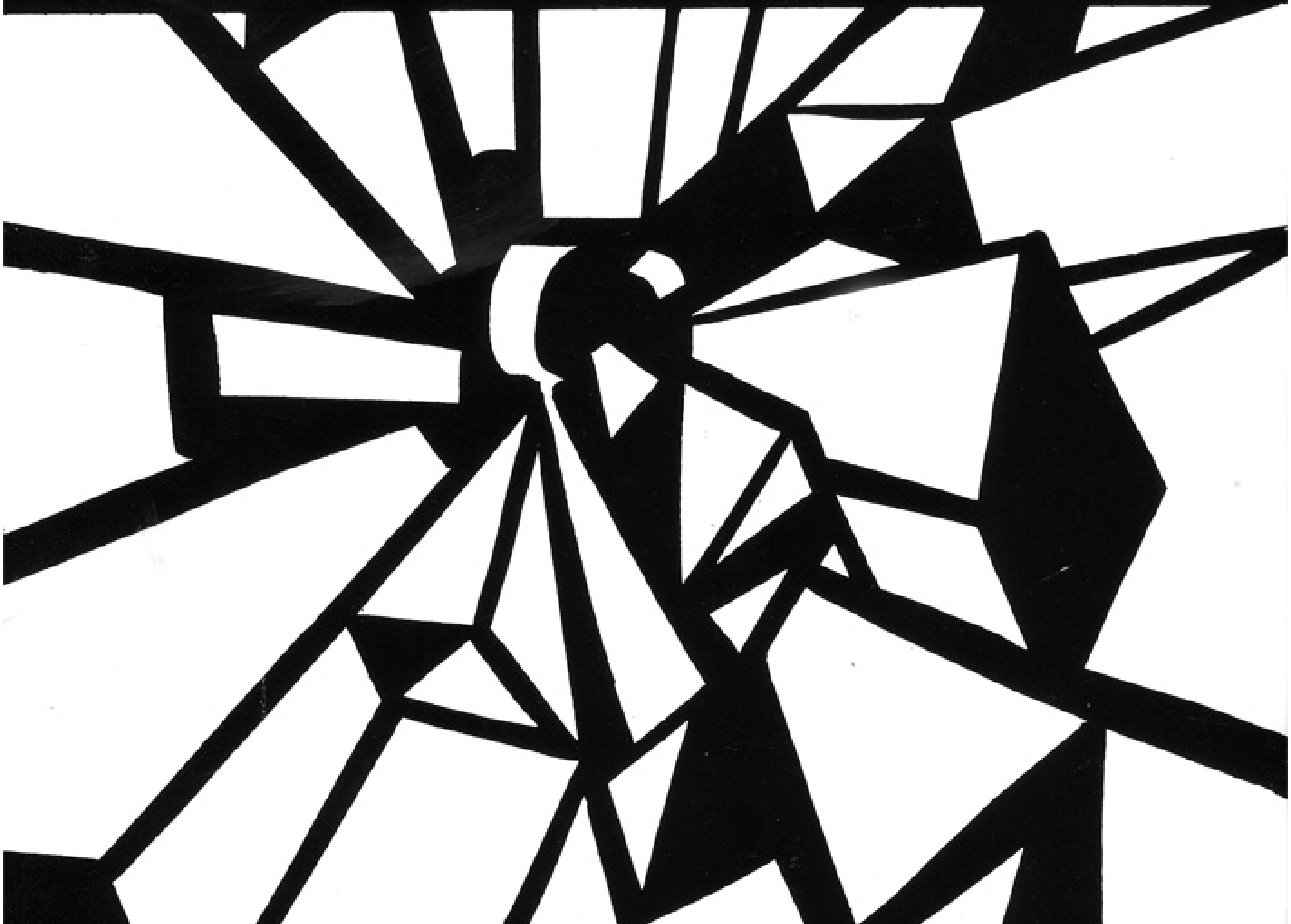
Volume 28 Number 5 1988

An International Journal

computers & structures

Editor-in-Chief HAROLD LIEBOWITZ



PERGAMON PRESS
Oxford • New York • Beijing • Frankfurt
São Paulo • Sydney • Tokyo • Toronto

Computers & Structures Vol. 28, No. 5, pp. 677–681, 1988
Printed in Great Britain.

0045-7949/88 \$3.00 + 0.00
Pergamon Press plc

TECHNICAL NOTE

ANALYTIC SOLUTIONS FOR TAPERED COLUMN BUCKLING

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(*Received* 28 *January* 1987)

Abstract—The energy method and symbolic aided analysis provide analytical expressions for the critical buckling load of tapered columns. The theory developed applies to any column whose cross sectional variation can be written as a function of the axial coordinate. Examples and percentage errata are included.

NOTATION

- B amplitude of assumed deflected shape
- $B(z)$ dimension of base of rectangular column cross section
- E Young's modulus for homogeneous material
- $H(z)$ dimension of height of rectangular column cross section
- H_1 height dimension at $z = 0$
- H_2 height dimension at $z = L$
- $I(z)$ moment of inertia about the centroid
- I_1 moment of inertia at $z = 0$
- I_2 moment of inertia at $z = L$
- k_i constant coefficient pertaining to moment of inertia model
- L length of the column before deflection
- $M(z)$ bending moment distribution
- P applied tip load
- P_{crn} critical buckling load
- $R(z)$ radius dimension of circular column cross section
- R_1 radius at $z = 0$
- R_2 radius at $z = L$
- U_c work done on column by tip load
- U_b strain energy due to bending
- $Y(z)$ assumed deflected shape
- λ vertical deflection of column

NONUNIFORM COLUMN ANALYSIS AS APPLIED TO STRINGER DESIGN FOR WING BAYS

Given a two-dimensional moment distribution corresponding to a wing loading (Fig. 1), the need arises to design stringers/columns for individual wing bays (Fig. 2) that can resist the variable moment at each cross section (i.e. the column should not buckle).

A traditional approach is to design the column to resist the highest bending moment (e.g. M_1) and have constant cross sectional area members for the bay. The weight penalty for such an approach can be prohibitive since at any location (other than where the largest moment occurs) the column is overdesigned. Alternatively, an optimal stringer design is one where the functional moment of inertia variation of the member differs from that of the moment distribution by a constant (i.e. column strength—and consequently the factor of safety—are both constant). A practical analysis and manufacturing compromise is to design the column such that the dimensions taper linearly between ribs.

STANDARD METHODS FOR PREDICTING CRITICAL BUCKLING LOADS WHERE THE CROSS SECTION OF THE COLUMN VARIES

Predicting the load which will cause the column to buckle is traditionally done by numerical or finite element techniques. An exact method is to solve the differential equation

$$EI(z)\frac{d^2Y(z)}{dz^2} + M(z) = 0, \tag{1}$$

with appropriate boundary conditions for the ends. Dinnik [1] shows that when the moment of inertia varies as a power, eqn (1) can be solved with Bessel functions where

$$I(z) = I_1\left(\frac{z}{a}\right)^n. \tag{2}$$

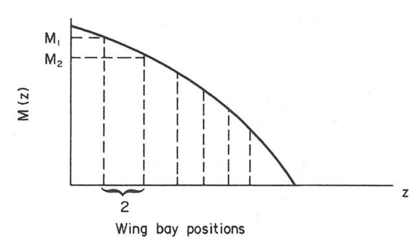

Fig. 1. Moment distribution as a function of wing length with wing bay positions indicated.

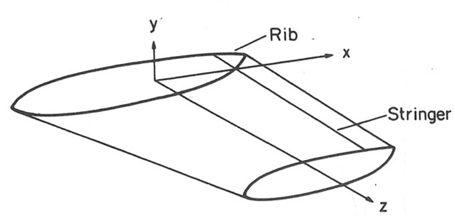

Fig. 2. Wing bay two corresponding to Fig. 1 moment distribution.

Aerospace structures such as T-stringers and I-beams are columns whose moment of inertia is not simply described by eqn (2) and require a more general expression associated with composite shapes of

$$I(z) = \sum_{i=0}^{n} K_i z^i, \tag{3}$$

where the K_is are constant coefficients dependent on the columns' cross section and by what order the dimensions taper.

CRITICAL BUCKLING LOAD WHEN I IS IN THE FORM OF EQN (3) USING THE ENERGY METHOD

A fixed–free column (Fig. 3) is used to demonstrate the present method. However, the technique is equally applicable for other boundary conditions.

Traditional expressions are established for work done by a compression and the resulting strain energy due to bending [3, pp. 82–88] where

$$U_c = P\lambda = P\,1/2\int_0^L \left(\frac{\mathrm{d}Y(z)}{\mathrm{d}z}\right)^2 \mathrm{d}z \tag{4}$$

$$U_b = \int_0^L \frac{(M(z))^2}{2EI(z)}\,\mathrm{d}z. \tag{5}$$

An assumed deflected shape that accounts for first mode buckling [3, p. 91] and satisfies the boundary conditions for a fixed–free column is then

$$Y(z) = B\left(1 - \cos\frac{\pi z}{2L}\right), \tag{6}$$

where $y = 0$ at $z = 0$ and $y = B$ at $z = L$.

Substituting

$$M(z) = -EI(z)\frac{\mathrm{d}^2 Y(z)}{\mathrm{d}z^2} \tag{7}$$

into eqn (5) and using eqn (6) yields

$$U_b = \frac{B^2 E\pi^4}{32L^4}\, I \int_0^L I(z)\cos^2\left(\frac{\pi z}{2L}\right)\mathrm{d}z. \tag{8}$$

The critical buckling condition occurs when $U_b = U_c$ and substituting eqn (3) for $I(z)$ in eqn (8) gives

$$P_{cr} = \frac{\pi^2 E}{2L^3}\int_0^L \sum_{i=0}^{n} K_i z^i \cos^2\left(\frac{\pi z}{2L}\right)\mathrm{d}z. \tag{9}$$

Similarly, for a pinned–pinned column

$$P_{cr} = \frac{2\pi^2 E}{L^3}\int_0^L \sum_{i=0}^{n} K_i z^i \sin^2\left(\frac{\pi z}{L}\right)\mathrm{d}z. \tag{10}$$

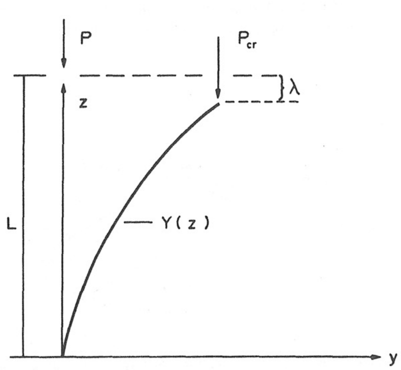

Fig. 3. A fixed–free column subjected to a tip load.

Example: Using the energy method, establish the critical buckling load for a fixed–free square pyramid or a truncated cone column (Fig. 4) whose moment of inertia varies about the x axis as $n = 4$ in eqn (3). This is a special case where, due to symmetry, the form of eqn (3) can be expressed in the form of eqn (2) and thus allows comparison between exact and energy method results.

Considering linear tapering dimensions for the square pyramid and the moment of inertia for a square cross section ($B = H$ in Fig. 4(a)),

$$H(z) = H_1 + \frac{zH_2}{L} - \frac{zH_1}{L} \tag{11}$$

$$I(z) = \frac{H(z)^4}{12} = \frac{\left(H_1 + \frac{zH_2}{L} - \frac{zH_1}{L}\right)^4}{12}. \tag{12}$$

Equation (12) is then expressed in the form of eqn (3) (i.e. $n = 4$),

$$I(z) = k_0 z^0 + k_1 z^1 + K_2 z^2 + k_3 z^3 + k_4 z^4, \tag{13}$$

where

$$K_0 = \frac{H_1^4}{12} \tag{14}$$

$$K_1 = \frac{H_1^3 H_2}{3L} - \frac{H_1^4}{3L} \tag{15}$$

$$K_2 = \frac{H_1^2 H_2^2}{2L^2} - \frac{H_1^3 H_2}{L^2}\frac{H_1^4}{2L^2} \tag{16}$$

$$K_3 = \frac{H_1 H_2^3}{3L^3} - \frac{H_1^2 H_2^2}{L^3} + \frac{H_1^3 H_2}{L^3} - \frac{H_1^4}{3L^3} \tag{17}$$

$$K_4 = \frac{H_2^4}{12L^4} - \frac{H_1 H_2^3}{3L^4} + \frac{H_1^2 H_2^2}{2L^4} - \frac{H_1^3 H_2}{3L^4} + \frac{H_1^4}{12L^4}. \tag{18}$$

Then, combining eqns (9) and (13)–(18),

$$\begin{aligned} P_{cr} = \frac{\pi^3 E}{6L^7}\int_0^L &((H_1^4 L^4 + (4H_1^3 H_2 - 4H_1^4)\,L^3 z \\ &+ (6H_1^2 H_2^2 - 12H_1^3 H_2 + 6H_1^4)\,L^2 z^2 \\ &+ (4H_1 H_2^3 - 12H_1^2 H_2^2 + 12H_1^3 H_2 \\ &- 4H_1^4)\,Lz^3 + (H_2^4 - 4H_1 H_2^3 + 6H_1^2 H_2^2 \\ &- 4H_1^3 H_2 + H_1^4)z^4)\cos^2\left(\frac{\pi z}{2L}\right)\mathrm{d}z. \end{aligned} \tag{19}$$

Substituting numerical constants for the moment of inertia ratios allows expression of H_1 in terms of H_2, or

$$H_1 = (I_1/I_2)^{1/4} H_2. \tag{20}$$

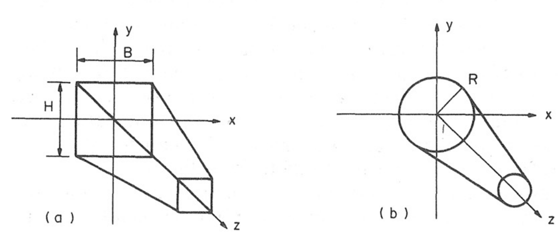

Fig. 4. Cross sections of columns for (a) square pyramid and (b) an axially symmetric truncated cone.

Then, combining eqn (20) and the solution to (19) (which is referenced as P_{cr5} in Appendix A) allows comparison (Fig. 5) between Dinnik's results and the energy method when the critical buckling load is expressed as

$$P_{cr} = \frac{mEI_1}{L^2}. \tag{21}$$

CONCLUSIONS

The example included is a special case where the form of eqn (3) can be expressed in the form of eqn (2). However, this technique is intended when form (2) does not apply as in the case of composite shapes (Appendix B). Notice the low percentage error even when only the first term is used in the series approximation for the deflected shape for the example.

In the past, the computations necessary to generate the forms in Appendices A and B would have tended to be prohibitive owing to their size. Symbolic analysis packages such as MuMath™ and MACSYMA™ allow for the generation and storage of such expressions in a more manageable manner. With the energy method and symbolic analysis it is only a matter of providing a functional description of the moment of inertia and the number of modes in order to generate the critical buckling load.

Notice that by using effective lengths, the range of applicability of the fixed–free and pinned–pinned cases can be extended to cover boundary conditions not discussed here.

Finally, it can be seen that for the sacrifice of expression size, an analytical solution can supersede the numerical.

Acknowledgement—Dr Nithiam T. Sivaneri is thanked for his critical comments and guidance.

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1. A. N. Dinnik, Design of columns of varying cross sections. *Trans. ASME* **51**, APM-51-11 (1929) and **54**, APM-54-16 (1932).
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APPENDIX A

The critical buckling formulas for the moment of inertia about the x-axis are given for five geometries whose dimensions taper linearly corresponding to Fig. 4. The odd number solutions are for the fixed–free case and the even number solutions are for the pinned–pinned case.

Case 1

$$B = B_1 + ZB_2L - ZB_1/L$$

$$H = \text{constant}$$

$$P_{cr1} = EH^3(4(-B_2 + B_1) + \pi^2(B_2 + B_1))/(96L^2)$$

$$P_{cr2} = \pi^2 EH^3(B_2 + B_1)/(24L^2).$$

Case 2

$$B = \text{constant}$$

$$H = H_1 + ZH_2/L - zH_1/L$$

$$P_{cr3} = BE(48(H_2(H_2(H_2 - 3H_1) + 3H_1^2) - H_1^3) + \pi^2(12(H_2(H_2(-H_2 + H_1) - H_1^2) + H_1^3) + \pi^2(H_2(H_2(H_2 + H_1) + H_1^2) + H_1^3)))/(192\pi^2L^2)$$

$$P_{cr4} = BE(3(-(H_2^3 + H_1^3) + H_2H_1(H_2 + H_1)) + \pi^2(H_2(H_2(H_2 + H_1) + H_1^2) + H_1^3))/(48L^2).$$

Case 3

$$B = H$$

$$H = H_1 + ZH_2/L - ZH_1/L$$

$$P_{cr5} = E(120(H_2(H_2^2(H_2 - 2H_1) + 2H_1^3) - H_1^4) + \pi^2(20(H_2(H_2^2(-H_2 + H_1) - H_1^3) + H_1^4) + \pi^2(H_2(H_2(H_2(H_2 + H_1) + H_1^2) + H_1^3) + H_1^4)))/(240\pi^2L^2)$$

$$P_{cr6} = E(15(H_2(H_2(H_2(H_2 - 4H_1) + 6H_1^2) - 4H_1^3) + H_1^4) + 2\pi^2(5(-(H_2^4 + H_1^4) + H_2H_1(H_2^2 + H_1^2)) + \pi^2(H_2(H_2(H_2(H_2 + H_1) + H_1^2) + H_1^3) + H_1^4)))/(120\pi^2L^2).$$

Case 4

$$R = R_1 + ZR_2/L - ZR_1/L$$

$$P_{cr7} = E(120(R_2(R_2^2(R_2 - 2R_1) + 2R_1^3) - R_1^4) + \pi^2(20(R_2(R_2^2(-R_2 + R_1) - R_1^3) + R_1^4) + \pi^2(R_2(R_2(R_2(R_2 + R_1) + R_1^2) + R_1^3) + R_1^4)))/(80\pi L^2)$$

$\frac{I_2}{I_1}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
m_1	---	1.202	1.505	1.710	1.870	2.002	2.116	2.217	2.308	2.391	$\frac{\pi^2}{4}$
m_2	0.886	1.500	1.684	1.824	1.942	2.047	2.142	2.237	2.314	2.392	$\frac{\pi^2}{4}$
Percent Error	--	24.8	11.9	6.7	3.8	2.2	1.2	0.6	0.2	0.1	0.0

Fig. 5. m_1 corresponds to Dinnik's results [3, p. 130] and m_2 corresponds to the energy method. The percent error is generated as the taper of the column becomes significant and this is due to the actual deflected shape no longer being a simple trigonometric shape as assumed.

$$P_{cr8} = E(15(R_2(R_2(R_2(R_2 - 4R_1) + 6R_1^2) - 4R_1^3) + R_1^4) + 2\pi^2(5(-(R_2^4 + R_1^4) + R_2R_1(R_2^2 + R_1^2)) + \pi^2(R_2(R_2(R_2(R_2 + R_1) + R_1^2) + R_1^3) + R_1^4)))/(40\pi L^2).$$

Case 5

$$B = B_1 + ZB_2/L - ZB_1/L$$

$$H = H_1 + ZH_2/L - ZH_1/L$$

$$P_{cr9} = E(240(H_2(H_2(H_2(2B_2 - B_1) - 3H_1B_2) + 3H_1^2B_1) + H_1^3(B_2 - 2B_1)) + \pi^2(20(H_2(H_2(H_2(-4B_2 + B_1) + 3H_1B_2) - 3H_1^2B_1) + H_1^3(-B_2 + 4B_1)) + \pi^2(H_2(H_2(H_2(4B_2 + B_1) + H_1(3B_2 + 2B_1)) + H_1^2(2B_2 + 3B_1)) + H_1^3(B_2 + 4B_1))))/(960\pi^2L^2)$$

$$P_{cr10} = E(30(B_1(H_1(H_1(H_1 - 3H_2) + 3H_2^2) - H_2^3) + B_2(H_1(H_1(-H_1 + 3H_2) - 3H_2^2) + H_2^3)) + \pi^2(5(B_1(H_1^2(-4H_1 + 3H_2) + H_2^3) + B_2(H_1(H_1^2 + 3H_2^2) - 4H_2^3)) + \pi^2(B_1(H_1(H_1(4H_1 + 3H_2) + 2H_2^2) + H_2^3) + B_2(H_1(H_1(H_1 + 2H_2) + 3H_2^2) + 4H_2^3))))/(240\,\pi^2L^2).$$

APPENDIX B

The critical buckling formula for the moment of inertia about the x-axis is given for a fixed–free I-beam with linearly tapering dimensions.

$$B(z) = B_1 - \frac{zB_1}{L} + \frac{zB_2}{L}$$

$$H(z) = H_1 - \frac{zH_1}{L} + \frac{zH_2}{L}$$

$$T(z) = T_1 - \frac{zT_1}{L} + \frac{zT_2}{L}$$

$$W(z) = W_1 - \frac{zW_1}{L} + \frac{zW_2}{L}$$

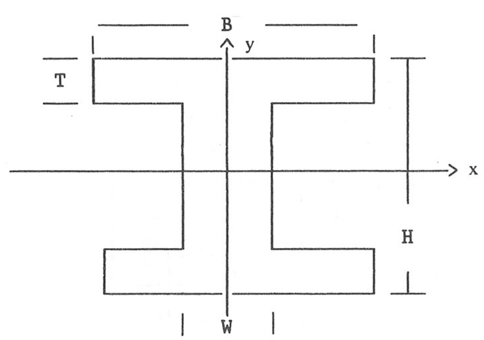

Fig. B1. An I-beam with B, H, T and W tapering linearly as a function of z, where $z = 0$ is the built-in end and $z = L$ is the free end.

$$I(z) = 2\left(\frac{B(z)T(z)^3}{12} + B(z)T(z)\left(\frac{H(z) - T(z)}{2}\right)^2 + \left(\frac{H(z)}{2} - T(z)\right)^3\frac{W(z)}{3}\right).$$

Combining $I(z)$ and the dimensional descriptions $B(z)$, $H(z)$, $T(z)$ and $W(z)$ with eqn (9) yields

$$P_{cr11} = -(((32\pi^4 - 640\pi^2 + 3840ET_2^3 + ((24\pi^4 + 480\pi^2 - 5760)ET_1 + (-48\pi^4 + 960\pi^2 - 5760EH_2 + (-12\pi^4 - 240\pi^2 + 2880)EH_1)T_2^2 + (16\pi^4ET_1^2 + ((-24\pi^4 - 480\pi^2 + 5760)EH_2 - 16\pi^4EH_1)T_1 + (24\pi^4 - 480\pi^2 + 2880)EH_2^2 + (12\pi^4 + 240\pi^2 - 2880)EH_1H_2 + 4\pi^4EH_1^2)T_2 + (8\pi^4 - 160\pi^2 + 1920)ET_1^3 + ((-12\pi^4 + 240\pi^2 - 2880)EH_1 - 8\pi^4EH_2)T_1^2 + ((6\pi^4 + 120\pi^2 - 1440)EH_2^2 + 8\pi^4EH_1H_2 + (6\pi^4 - 120\pi^2 + 1440)EH_1^2)T_1 + (-4\pi^4 + 80\pi^2 - 480)EH_2^3 + (-3\pi^4 - 60\pi^2 + 720)EH_1H_2^2 - 2\pi^4EH_1^2H_2 + (-\pi^4 + 20\pi^2 - 240)EH_1^3)W_2 + ((8\pi^4 + 160\pi^2 - 1920)ET_2^3 + (16\pi^4ET_1 + (-12\pi^4 - 240\pi^2 + 2880)EH_2 - 8\pi^4EH_1)T_2^2 + ((24\pi^4 - 480\pi^2 + 5760)ET_1^2 + ((-24\pi^4 + 480\pi^2 - 5760)EH_1 - 16\pi^4EH_2)T_1(6\pi^4 + 120\pi^2 - 1440)EH_2^2 + 8\pi^4EH_1H_2 + (6\pi^4 - 120\pi^2 + 1440)EH_1^2)T_2 + (32\pi^4 + 640\pi^2 - 3840)ET_1^3 + ((-12\pi^4 + 240\pi^2 - 2880)EH_2 + (-48\pi^4 - 960\pi^2 + 5760)EH_1)T_1^2 + (4\pi^4EH_2^2 + (12\pi^4 - 240\pi^2 + 2880)EH_1H_2 + (24\pi^4 + 480\pi^2 - 2880)EH_1^2)T_1 + (-\pi^4 - 20\pi^2 + 240)EH_2^3 - 2\pi^4EH_1H_2^2 + (-3\pi^4 + 60\pi^2 - 720)EH_1^2H_2 + (-4\pi^4 - 80\pi^2 + 480)EH_1^3)W_1 + ((-32\pi^4 + 640\pi^2 - 3840)B_2 + (-8\pi^4 - 160\pi^2 + 1920)B_1)ET_2^3 + (((-24\pi^4 - 480\pi^2 + 5760)B_2 - 16\pi^4B_1)ET_1 + ((48\pi^4 - 960\pi^2 + 5760)B_2 + (12\pi^4 + 240\pi^2 - 2880)B_1)EH_2 + ((12\pi^4 + 240\pi^2 - 2880)B_2 + 8\pi^4B_1)EH_1)T_2^2 + (((-24\pi^4 + 480\pi^2 - 5760)B_1 - 16\pi^4B_2)ET_1^2 + (((24\pi^4 + 480\pi^2 - 5760)B_2 + 16\pi^4B_1)EH_2 + (16\pi^4B_2 + (24\pi^4 - 480\pi^2 + 5760)B_1)EH_1)T_1 + ((-24\pi^4 + 480\pi^2 - 2880)B_2 + (-6\pi^4 - 120\pi^2 + 1440)B_1)EH_2^2 + ((-12\pi^4 - 240\pi^2 + 2880)B_2$$

$$- 8\pi^4 B_1)EH_1H_2 + ((-6\pi^4 + 120\pi^2 - 1440)B_1$$
$$- 4\pi^4 B_2)EH_1^2)T_2 + ((-8\pi^4 + 160\pi^2 - 1920)B_2$$
$$+ (-32\pi^4 - 640\pi^2 + 3840)B_1)ET_1^3$$
$$+ ((8\pi^4 B_2 + (12\pi^4 - 240\pi^2 + 2880)B_1)EH_2$$
$$+ ((12\pi^4 - 240\pi^2 + 2880)B_2$$
$$+ (48\pi^4 + 960\pi^2 - 5760)B_1)EH_1)T_1^2$$
$$+ (((-6\pi^4 - 120\pi^2 + 1440)B_2$$
$$- 4\pi^4 B_1)EH_2^2 + ((-12\pi^4 + 240\pi^2 - 2880)B_1$$
$$- 8\pi^4 B_2)EH_1H_2 + ((-6\pi^4 + 120\pi^2 - 1440)B_2$$
$$+ (-24\pi^4 - 480\pi^2 + 2880)B_1)EH_1^2$$
$$+ T_1)/(960\pi^2 l^2).$$